# Higher Modular Representations of Lie Algebras

Matt Westaway

Colloque Tournant 2021 du GDR Théorie de Lie Algébrique et Géométrique

24th March 2021

# Algebraic Groups

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, e.g. $G = \mathrm{GL}_n$, $n \times n$ invertible matrices.

# Algebraic Groups

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, e.g. $G = \mathrm{GL}_n$, $n \times n$ invertible matrices.

Let $T$ be a maximal torus of $G$, e.g. $T =$ invertible diagonal matrices.

# Algebraic Groups

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, e.g. $G = \mathrm{GL}_n$, $n \times n$ invertible matrices.

Let $T$ be a maximal torus of $G$, e.g. $T =$ invertible diagonal matrices.

Let $B$ be a Borel subgroup of $G$ containing $T$, e.g. $B =$ invertible upper triangular matrices.

# Algebraic Groups

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, e.g. $G = \mathrm{GL}_n$, $n \times n$ invertible matrices.

Let $T$ be a maximal torus of $G$, e.g. $T =$ invertible diagonal matrices.

Let $B$ be a Borel subgroup of $G$ containing $T$, e.g. $B =$ invertible upper triangular matrices.

We write $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{h} = \mathrm{Lie}(T)$ (e.g. $\mathfrak{g} = M_n(\mathbb{K})$, $\mathfrak{b} =$ upper triangular matrices, $\mathfrak{h} =$ diagonal matrices).

# Algebraic Groups

Let $G$ be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$, e.g. $G = \mathrm{GL}_n$, $n \times n$ invertible matrices.

Let $T$ be a maximal torus of $G$, e.g. $T =$ invertible diagonal matrices.

Let $B$ be a Borel subgroup of $G$ containing $T$, e.g. $B =$ invertible upper triangular matrices.

We write $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{h} = \mathrm{Lie}(T)$ (e.g. $\mathfrak{g} = M_n(\mathbb{K})$, $\mathfrak{b} =$ upper triangular matrices, $\mathfrak{h} =$ diagonal matrices).

Let $\Phi$ be the root system of $G$, $\Phi^+$ the positive roots, $\Pi$ the simple roots.

# Differentiation of Modules

Let $M$ be a finite-dimensional module over the algebraic group $G$.

# Differentiation of Modules

Let $M$ be a finite-dimensional module over the algebraic group $G$.

Differentiating gives a module $M$ over the Lie algebra $\mathfrak{g}$.

# Differentiation of Modules

Let $M$ be a finite-dimensional module over the algebraic group $G$.

Differentiating gives a module $M$ over the Lie algebra $\mathfrak{g}$.

Recall that elements of $\mathfrak{g}$ are derivations $\delta : \mathbb{K}[G] \to \mathbb{K}[G]$.

# Differentiation of Modules

Let $M$ be a finite-dimensional module over the algebraic group $G$.

Differentiating gives a module $M$ over the Lie algebra $\mathfrak{g}$.

Recall that elements of $\mathfrak{g}$ are derivations $\delta : \mathbb{K}[G] \to \mathbb{K}[G]$.

In general, if $\delta, \mu$ are derivations, $\delta \circ \mu$ need not be. But $\underbrace{\delta \circ \delta \circ \cdots \circ \delta}_{p}$ is!

# Differentiation of Modules

Let $M$ be a finite-dimensional module over the algebraic group $G$.

Differentiating gives a module $M$ over the Lie algebra $\mathfrak{g}$.

Recall that elements of $\mathfrak{g}$ are derivations $\delta : \mathbb{K}[G] \to \mathbb{K}[G]$.

In general, if $\delta, \mu$ are derivations, $\delta \circ \mu$ need not be. But $\underbrace{\delta \circ \delta \circ \cdots \circ \delta}_{p}$ is!

This gives a map $\mathfrak{g} \to \mathfrak{g}$, $\delta \mapsto \delta^{[p]}$ satisfying certain properties.

# Reduced Enveloping Algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, so

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle}.$$

# Reduced Enveloping Algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, so

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle}.$$

We look at the **restricted enveloping algebra**

$$U_0(\mathfrak{g}) = \frac{U(\mathfrak{g})}{\langle \delta^p - \delta^{[p]} \mid \delta \in \mathfrak{g} \rangle}$$

where $\delta^p$ is the $p$-th power in $U(\mathfrak{g})$.

# Reduced Enveloping Algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, so

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle}.$$

We look at the **restricted enveloping algebra**

$$U_0(\mathfrak{g}) = \frac{U(\mathfrak{g})}{\langle \delta^p - \delta^{[p]} \mid \delta \in \mathfrak{g} \rangle}$$

where $\delta^p$ is the $p$-th power in $U(\mathfrak{g})$.
Each $\mathfrak{g}$-module obtained from a $G$-module is a $U_0(\mathfrak{g})$-module. But not all $\mathfrak{g}$-modules are obtained in this way.

# Reduced Enveloping Algebras

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, so

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x, y] \,|\, x, y \in \mathfrak{g} \rangle}.$$

We look at the **restricted enveloping algebra**

$$U_0(\mathfrak{g}) = \frac{U(\mathfrak{g})}{\langle \delta^p - \delta^{[p]} \,|\, \delta \in \mathfrak{g} \rangle}$$

where $\delta^p$ is the $p$-th power in $U(\mathfrak{g})$.
Each $\mathfrak{g}$-module obtained from a $G$-module is a $U_0(\mathfrak{g})$-module. But not all $\mathfrak{g}$-modules are obtained in this way.
Given $\chi \in \mathfrak{g}^*$, we define the **reduced enveloping algebra**

$$U_\chi(\mathfrak{g}) = \frac{U(\mathfrak{g})}{\langle \delta^p - \delta^{[p]} - \chi(\delta)^p \,|\, \delta \in \mathfrak{g} \rangle}.$$

# Properties

## Proposition

1. *The map* $\xi : \mathfrak{g} \to U(\mathfrak{g})$ *which sends* $\delta \mapsto \delta^p - \delta^{[p]}$ *is semilinear and has central image.*

# Properties

## Proposition

1. *The map* $\xi : \mathfrak{g} \to U(\mathfrak{g})$ *which sends* $\delta \mapsto \delta^p - \delta^{[p]}$ *is semilinear and has central image.*
2. *Every irreducible* $\mathfrak{g}$*-module is finite-dimensional.*

# Properties

## Proposition

1. *The map* $\xi : \mathfrak{g} \to U(\mathfrak{g})$ *which sends* $\delta \mapsto \delta^p - \delta^{[p]}$ *is semilinear and has central image.*
2. *Every irreducible* $\mathfrak{g}$*-module is finite-dimensional.*
3. *Every irreducible* $\mathfrak{g}$*-module is an irreducible* $U_\chi(\mathfrak{g})$*-module for some* $\chi \in \mathfrak{g}^*$*.*

# Properties

## Proposition

1. *The map* $\xi : \mathfrak{g} \to U(\mathfrak{g})$ *which sends* $\delta \mapsto \delta^p - \delta^{[p]}$ *is semilinear and has central image.*
2. *Every irreducible* $\mathfrak{g}$*-module is finite-dimensional.*
3. *Every irreducible* $\mathfrak{g}$*-module is an irreducible* $U_\chi(\mathfrak{g})$*-module for some* $\chi \in \mathfrak{g}^*$*.*
4. *Given* $g \in G$ *and* $\chi \in \mathfrak{g}^*$*, the algebras* $U_\chi(\mathfrak{g})$ *and* $U_{g\cdot\chi}(\mathfrak{g})$ *are isomorphic.*

# Properties

## Proposition

1. *The map* $\xi : \mathfrak{g} \to U(\mathfrak{g})$ *which sends* $\delta \mapsto \delta^p - \delta^{[p]}$ *is semilinear and has central image.*
2. *Every irreducible* $\mathfrak{g}$*-module is finite-dimensional.*
3. *Every irreducible* $\mathfrak{g}$*-module is an irreducible* $U_\chi(\mathfrak{g})$*-module for some* $\chi \in \mathfrak{g}^*$*.*
4. *Given* $g \in G$ *and* $\chi \in \mathfrak{g}^*$*, the algebras* $U_\chi(\mathfrak{g})$ *and* $U_{g\cdot\chi}(\mathfrak{g})$ *are isomorphic.*
5. *Each algebra* $U_\chi(\mathfrak{g})$ *is finite-dimensional, of dimension* $p^{\dim \mathfrak{g}}$*.*

# Triangular Decomposition

For $\alpha \in \Phi$, we write $e_\alpha$ for a chosen root vector in $\mathfrak{g}_\alpha$. Write $\mathfrak{n}^-$ for the Lie subalgebra generated by $e_{-\alpha}$ for $\alpha \in \Phi^+$, and $\mathfrak{n}^+$ for the Lie subalgebra generated by $e_\alpha$ for $\alpha \in \Phi^+$.

# Triangular Decomposition

For $\alpha \in \Phi$, we write $e_\alpha$ for a chosen root vector in $\mathfrak{g}_\alpha$. Write $\mathfrak{n}^-$ for the Lie subalgebra generated by $e_{-\alpha}$ for $\alpha \in \Phi^+$, and $\mathfrak{n}^+$ for the Lie subalgebra generated by $e_\alpha$ for $\alpha \in \Phi^+$.

We then have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

# Triangular Decomposition

For $\alpha \in \Phi$, we write $e_\alpha$ for a chosen root vector in $\mathfrak{g}_\alpha$. Write $\mathfrak{n}^-$ for the Lie subalgebra generated by $e_{-\alpha}$ for $\alpha \in \Phi^+$, and $\mathfrak{n}^+$ for the Lie subalgebra generated by $e_\alpha$ for $\alpha \in \Phi^+$.

We then have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Note that $e_\alpha^{[p]} = 0$ for all $\alpha \in \Phi$, while $\mathfrak{h}$ has a basis $h_1, \ldots, h_d$ with $h_i^{[p]} = h_i$ for all $1 \leqslant i \leqslant d$.

# Triangular Decomposition

For $\alpha \in \Phi$, we write $e_\alpha$ for a chosen root vector in $\mathfrak{g}_\alpha$. Write $\mathfrak{n}^-$ for the Lie subalgebra generated by $e_{-\alpha}$ for $\alpha \in \Phi^+$, and $\mathfrak{n}^+$ for the Lie subalgebra generated by $e_\alpha$ for $\alpha \in \Phi^+$.

We then have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Note that $e_\alpha^{[p]} = 0$ for all $\alpha \in \Phi$, while $\mathfrak{h}$ has a basis $h_1, \ldots, h_d$ with $h_i^{[p]} = h_i$ for all $1 \leqslant i \leqslant d$.

We will only consider $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$.

# Baby Verma Modules

Let $\lambda \in \Lambda_\chi$, where

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}.$$

We define $\mathbb{K}_\lambda$ to be the 1-dimensional $U_\chi(\mathfrak{b})$-module on which $\mathfrak{n}^+$ acts trivially and $\mathfrak{h}$ acts via $\lambda$.

# Baby Verma Modules

Let $\lambda \in \Lambda_\chi$, where

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}.$$

We define $\mathbb{K}_\lambda$ to be the 1-dimensional $U_\chi(\mathfrak{b})$-module on which $\mathfrak{n}^+$ acts trivially and $\mathfrak{h}$ acts via $\lambda$.

We then define the **baby Verma module**

$$Z_\chi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{K}_\lambda.$$

# Baby Verma Modules

Let $\lambda \in \Lambda_\chi$, where

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}.$$

We define $\mathbb{K}_\lambda$ to be the 1-dimensional $U_\chi(\mathfrak{b})$-module on which $\mathfrak{n}^+$ acts trivially and $\mathfrak{h}$ acts via $\lambda$.

We then define the **baby Verma module**

$$Z_\chi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{K}_\lambda.$$

**Proposition**

*Every irreducible $U_\chi(\mathfrak{g})$-module is a quotient of some baby Verma module.*

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.
Recall that $\mathfrak{sl}_2$ has $\mathbb{K}$-basis $e, h, f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.
Recall that $\mathfrak{sl}_2$ has $\mathbb{K}$-basis $e, h, f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Each $\chi \in \mathfrak{g}^*$ is conjugate to a linear form of one of the three following types:

1. $\chi = 0$.

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.
Recall that $\mathfrak{sl}_2$ has $\mathbb{K}$-basis $e, h, f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Each $\chi \in \mathfrak{g}^*$ is conjugate to a linear form of one of the three following types:

1. $\chi = 0$.
2. $e \mapsto 0, f \mapsto 0, h \mapsto t$ for some $t \in \mathbb{K}^*$. We call such $\chi$ **semisimple**.

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.
Recall that $\mathfrak{sl}_2$ has $\mathbb{K}$-basis $e, h, f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Each $\chi \in \mathfrak{g}^*$ is conjugate to a linear form of one of the three following types:

1. $\chi = 0$.
2. $e \mapsto 0, f \mapsto 0, h \mapsto t$ for some $t \in \mathbb{K}^*$. We call such $\chi$ **semisimple**.
3. $e \mapsto 0, f \mapsto 1, h \mapsto 0$. We call such $\chi$ **nilpotent**.

# $\mathrm{SL}_2$

When $G = \mathrm{SL}_2$, and so $\mathfrak{g} = \mathfrak{sl}_2$, we classify the irreducible $\mathfrak{g}$-modules when $p > 2$.
Recall that $\mathfrak{sl}_2$ has $\mathbb{K}$-basis $e, h, f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Each $\chi \in \mathfrak{g}^*$ is conjugate to a linear form of one of the three following types:

1. $\chi = 0$.
2. $e \mapsto 0, f \mapsto 0, h \mapsto t$ for some $t \in \mathbb{K}^*$. We call such $\chi$ **semisimple**.
3. $e \mapsto 0, f \mapsto 1, h \mapsto 0$. We call such $\chi$ **nilpotent**.

We may identify $\mathfrak{h}^*$ with $\mathbb{K}$. When $\chi = 0$ or $\chi$ is nilpotent, we have $\Lambda_\chi = \mathbb{F}_p$. When $\chi$ is semisimple we have $\Lambda_\chi \subseteq \mathbb{K} \backslash \mathbb{F}_p$.

# $\mathrm{SL}_2$

**Theorem (Block '62, Rudakov-Shafarevich '67)**

1. *Suppose* $\chi = 0$. *Then there exists exactly one irreducible* $U_\chi(\mathfrak{g})$*-module of each dimension* $1, \ldots, p$*, and each irreducible* $U_\chi(\mathfrak{g})$ *appears in this way.*

# $\mathrm{SL}_2$

**Theorem (Block '62, Rudakov-Shafarevich '67)**

1. *Suppose* $\chi = 0$. *Then there exists exactly one irreducible* $U_\chi(\mathfrak{g})$*-module of each dimension* $1, \ldots, p$*, and each irreducible* $U_\chi(\mathfrak{g})$ *appears in this way.*
2. *Suppose* $\chi$ *is non-zero semisimple. Then each baby Verma module* $Z_\chi(\lambda)$*, for* $\lambda \in \Lambda_\chi$*, is irreducible. Each irreducible module appears in this way. Furthermore,* $Z_\chi(\lambda) \cong Z_\chi(\mu)$ *if and only if* $\lambda = \mu$*. So there are exactly* $p$ *non-isomorphic irreducible* $U_\chi(\mathfrak{g})$*-modules.*

# $\mathrm{SL}_2$

**Theorem (Block '62, Rudakov-Shafarevich '67)**

1. *Suppose* $\chi = 0$. *Then there exists exactly one irreducible* $U_\chi(\mathfrak{g})$*-module of each dimension* $1, \ldots, p$, *and each irreducible* $U_\chi(\mathfrak{g})$ *appears in this way.*
2. *Suppose* $\chi$ *is non-zero semisimple. Then each baby Verma module* $Z_\chi(\lambda)$, *for* $\lambda \in \Lambda_\chi$, *is irreducible. Each irreducible module appears in this way. Furthermore,* $Z_\chi(\lambda) \cong Z_\chi(\mu)$ *if and only if* $\lambda = \mu$. *So there are exactly* $p$ *non-isomorphic irreducible* $U_\chi(\mathfrak{g})$*-modules.*
3. *Suppose* $\chi$ *is non-zero nilpotent. Then each baby Verma module* $Z_\chi(\lambda)$, *for* $\lambda \in \Lambda_\chi$, *is irreducible. Each irreducible module appears in this way. Furthermore,* $Z_\chi(\lambda) \cong Z_\chi(\mu)$ *if and only if* $\lambda = \mu$ *or* $\lambda = p - \mu - 2$. *So there are exactly* $\frac{p+1}{2}$ *non-isomorphic irreducible* $U_\chi(\mathfrak{g})$*-modules.*

# Coordinate Algebra

Recall that $\mathbb{K}[G]$ is the coordinate algebra of $G$. This is a Hopf algebra, with comultiplication, counit and antipode:

$$\Delta : \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G] = \mathbb{K}[G \times G], \quad \phi \mapsto ((g_1, g_2) \mapsto \phi(g_1 g_2)),$$

$$\varepsilon : \mathbb{K}[G] \to \mathbb{K}, \qquad \phi \mapsto \phi(1_G),$$

$$S : \mathbb{K}[G] \to \mathbb{K}[G], \qquad \phi \mapsto (g \mapsto \phi(g^{-1})).$$

# Coordinate Algebra

Recall that $\mathbb{K}[G]$ is the coordinate algebra of $G$. This is a Hopf algebra, with comultiplication, counit and antipode:

$$\Delta : \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G] = \mathbb{K}[G \times G], \quad \phi \mapsto ((g_1, g_2) \mapsto \phi(g_1 g_2)),$$

$$\varepsilon : \mathbb{K}[G] \to \mathbb{K}, \qquad \phi \mapsto \phi(1_G),$$

$$S : \mathbb{K}[G] \to \mathbb{K}[G], \qquad \phi \mapsto (g \mapsto \phi(g^{-1})).$$

We let $I$ be the augmentation ideal of $\mathbb{K}[G]$, i.e.

$$I = \ker(\varepsilon).$$

# Distribution Algebra

The **distribution algebra** $\mathrm{Dist}(G)$ of $G$ is a filtered Hopf algebra

$$\mathrm{Dist}(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n(G)$$

where $\mathrm{Dist}_n(G)$ is a $\mathbb{K}$-vector space (in fact coalgebra) defined as

$$\mathrm{Dist}_n(G) := \{\delta : \mathbb{K}[G] \to \mathbb{K} \mid \delta \text{ is linear and } \delta(I^{n+1}) = 0\}.$$

# Distribution Algebra

The **distribution algebra** $\mathrm{Dist}(G)$ of $G$ is a filtered Hopf algebra

$$\mathrm{Dist}(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n(G)$$

where $\mathrm{Dist}_n(G)$ is a $\mathbb{K}$-vector space (in fact coalgebra) defined as

$$\mathrm{Dist}_n(G) := \{\delta : \mathbb{K}[G] \to \mathbb{K} \mid \delta \text{ is linear and } \delta(I^{n+1}) = 0\}.$$

Given $\delta, \mu \in \mathrm{Dist}(G)$, we have the product $\delta\mu$ defined as the composition

$$\mathbb{K}[G] \xrightarrow{\Delta} \mathbb{K}[G] \otimes \mathbb{K}[G] \xrightarrow{\delta \otimes \mu} \mathbb{K} \otimes \mathbb{K} \xrightarrow{\sim} \mathbb{K}.$$

# Distribution Algebra

The **distribution algebra** $\mathrm{Dist}(G)$ of $G$ is a filtered Hopf algebra

$$\mathrm{Dist}(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n(G)$$

where $\mathrm{Dist}_n(G)$ is a $\mathbb{K}$-vector space (in fact coalgebra) defined as

$$\mathrm{Dist}_n(G) := \{\delta : \mathbb{K}[G] \to \mathbb{K} \mid \delta \text{ is linear and } \delta(I^{n+1}) = 0\}.$$

Given $\delta, \mu \in \mathrm{Dist}(G)$, we have the product $\delta\mu$ defined as the composition

$$\mathbb{K}[G] \xrightarrow{\Delta} \mathbb{K}[G] \otimes \mathbb{K}[G] \xrightarrow{\delta \otimes \mu} \mathbb{K} \otimes \mathbb{K} \xrightarrow{\sim} \mathbb{K}.$$

The unit of $\mathrm{Dist}(G)$ is the counit $\varepsilon$ of $\mathbb{K}[G]$.

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

## Proposition

*Let* $\delta \in \mathrm{Dist}_n(G)$, $\mu \in \mathrm{Dist}_m(G)$. *Then*

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

## Proposition

*Let* $\delta \in \mathrm{Dist}_n(G)$, $\mu \in \mathrm{Dist}_m(G)$. *Then*

1. *The product* $\delta\mu$ *lies in* $\mathrm{Dist}_{n+m}(G)$.

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

## Proposition

*Let* $\delta \in \mathrm{Dist}_n(G)$, $\mu \in \mathrm{Dist}_m(G)$. *Then*

1. *The product* $\delta\mu$ *lies in* $\mathrm{Dist}_{n+m}(G)$.
2. *The commutator* $[\delta, \mu] = \delta\mu - \mu\delta$ *lies in* $\mathrm{Dist}_{n+m-1}(G)$.

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

## Proposition

*Let* $\delta \in \mathrm{Dist}_n(G)$, $\mu \in \mathrm{Dist}_m(G)$. *Then*

1. *The product* $\delta\mu$ *lies in* $\mathrm{Dist}_{n+m}(G)$.
2. *The commutator* $[\delta, \mu] = \delta\mu - \mu\delta$ *lies in* $\mathrm{Dist}_{n+m-1}(G)$.
3. *If, furthermore,* $\delta \in \mathrm{Dist}_n^+(G)$ *and* $\mu \in \mathrm{Dist}_m^+(G)$ *then* $\delta\mu \in \mathrm{Dist}_{n+m}^+(G)$ *and* $[\delta, \mu] \in \mathrm{Dist}_{m+n-1}^+(G)$.

# Distribution Algebra

We furthermore define

$$\mathrm{Dist}_n^+(G) := \{\delta \in \mathrm{Dist}_n(G) \mid \delta(1_{\mathbb{K}[G]}) = 0\}$$

and

$$\mathrm{Dist}^+(G) = \bigcup_{n \geqslant 0} \mathrm{Dist}_n^+(G).$$

## Proposition

*Let* $\delta \in \mathrm{Dist}_n(G)$, $\mu \in \mathrm{Dist}_m(G)$. *Then*

1. *The product* $\delta\mu$ *lies in* $\mathrm{Dist}_{n+m}(G)$.
2. *The commutator* $[\delta, \mu] = \delta\mu - \mu\delta$ *lies in* $\mathrm{Dist}_{n+m-1}(G)$.
3. *If, furthermore,* $\delta \in \mathrm{Dist}_n^+(G)$ *and* $\mu \in \mathrm{Dist}_m^+(G)$ *then* $\delta\mu \in \mathrm{Dist}_{n+m}^+(G)$ *and* $[\delta, \mu] \in \mathrm{Dist}_{m+n-1}^+(G)$.

*In particular,* $\mathrm{Dist}_1^+(G) = \mathfrak{g}$.

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

The **first Frobenius kernel** $G_1$ of $G$ is then defined as the kernel of this map.

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

The **first Frobenius kernel** $G_1$ of $G$ is then defined as the kernel of this map.
We may define $\mathrm{Dist}(G_1)$ in the same way we defined $\mathrm{Dist}(G)$.

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

The **first Frobenius kernel** $G_1$ of $G$ is then defined as the kernel of this map.
We may define $\mathrm{Dist}(G_1)$ in the same way we defined $\mathrm{Dist}(G)$.
There exists a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Dist}(G)^{(-)}$, which extends to an algebra homomorphism

$$U(\mathfrak{g}) \to \mathrm{Dist}(G).$$

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

The **first Frobenius kernel** $G_1$ of $G$ is then defined as the kernel of this map.
We may define $\mathrm{Dist}(G_1)$ in the same way we defined $\mathrm{Dist}(G)$.
There exists a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Dist}(G)^{(-)}$, which extends to an algebra homomorphism

$$U(\mathfrak{g}) \to \mathrm{Dist}(G).$$

When the characteristic of $\mathbb{K}$ is zero, this is an isomorphism.

# Frobenius kernels

The **Frobenius morphism**

$$F : G \to G^{(1)}$$

is obtained from the comorphism

$$F^* : \mathbb{K}[G]^{(-1)} \to \mathbb{K}[G], \qquad f \mapsto f^p.$$

The **first Frobenius kernel** $G_1$ of $G$ is then defined as the kernel of this map.
We may define $\mathrm{Dist}(G_1)$ in the same way we defined $\mathrm{Dist}(G)$.
There exists a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Dist}(G)^{(-)}$, which extends to an algebra homomorphism

$$U(\mathfrak{g}) \to \mathrm{Dist}(G).$$

When the characteristic of $\mathbb{K}$ is zero, this is an isomorphism.
But, when the characteristic of $\mathbb{K}$ is $p > 0$, it is generally neither injective or surjective.

# Friedlander-Parshall Question

## Theorem

*In characteristic $p > 0$, the algebra homomorphism passes to an algebra isomorphism*

$$U_0(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G_1).$$

# Friedlander-Parshall Question

**Theorem**

*In characteristic $p > 0$, the algebra homomorphism passes to an algebra isomorphism*

$$U_0(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G_1).$$

By iterating the Frobenius map, we may instead take the kernel of

$$F^r : G \to G^{(r)}.$$

# Friedlander-Parshall Question

**Theorem**

*In characteristic $p > 0$, the algebra homomorphism passes to an algebra isomorphism*

$$U_0(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G_1).$$

By iterating the Frobenius map, we may instead take the kernel of

$$F^r : G \to G^{(r)}.$$

We call it a **higher Frobenius kernel** of $G$ and denote it by $G_r$.

# Friedlander-Parshall Question

**Theorem**

*In characteristic $p > 0$, the algebra homomorphism passes to an algebra isomorphism*

$$U_0(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G_1).$$

By iterating the Frobenius map, we may instead take the kernel of

$$F^r : G \to G^{(r)}.$$

We call it a **higher Frobenius kernel** of $G$ and denote it by $G_r$.

**Question (Friedlander-Parshall, 1990)**

*Do the reduced enveloping algebras $U_\chi(\mathfrak{g})$ have natural analogues corresponding to the infinitesimal group schemes $G_r$ associated to $G$ for $r > 1$?*

# Higher Universal Enveloping Algebras

We define the **higher universal enveloping algebra** of degree $r \in \mathbb{N}$ as

$$U^{[r]}(G) := \frac{T(\mathrm{Dist}^{+}_{p^{r+1}-1}(G))}{Q}$$

where $Q$ is the two-sided ideal generated by the relations:

# Higher Universal Enveloping Algebras

We define the **higher universal enveloping algebra** of degree $r \in \mathbb{N}$ as

$$U^{[r]}(G) := \frac{T(\mathrm{Dist}^{+}_{p^{r+1}-1}(G))}{Q}$$

where $Q$ is the two-sided ideal generated by the relations:

- $\delta \otimes \mu - \delta\mu$ for $\delta \in \mathrm{Dist}^{+}_{i}(G)$, $\mu \in \mathrm{Dist}^{+}_{j}(G)$ with $i + j < p^{r+1}$, and

# Higher Universal Enveloping Algebras

We define the **higher universal enveloping algebra** of degree $r \in \mathbb{N}$ as

$$U^{[r]}(G) := \frac{T(\mathrm{Dist}^+_{p^{r+1}-1}(G))}{Q}$$

where $Q$ is the two-sided ideal generated by the relations:

- $\delta \otimes \mu - \delta\mu$ for $\delta \in \mathrm{Dist}^+_i(G)$, $\mu \in \mathrm{Dist}^+_j(G)$ with $i + j < p^{r+1}$, and
- $\delta \otimes \mu - \mu \otimes \delta - [\delta, \mu]$ for $\delta \in \mathrm{Dist}^+_i(G)$, $\mu \in \mathrm{Dist}^+_j(G)$ with $i + j \leqslant p^{r+1}$.

# Higher Universal Enveloping Algebras

We define the **higher universal enveloping algebra** of degree $r \in \mathbb{N}$ as

$$U^{[r]}(G) := \frac{T(\mathrm{Dist}^{+}_{p^{r+1}-1}(G))}{Q}$$

where $Q$ is the two-sided ideal generated by the relations:

- $\delta \otimes \mu - \delta\mu$ for $\delta \in \mathrm{Dist}^{+}_{i}(G)$, $\mu \in \mathrm{Dist}^{+}_{j}(G)$ with $i + j < p^{r+1}$, and
- $\delta \otimes \mu - \mu \otimes \delta - [\delta, \mu]$ for $\delta \in \mathrm{Dist}^{+}_{i}(G)$, $\mu \in \mathrm{Dist}^{+}_{j}(G)$ with $i + j \leqslant p^{r+1}$.

Here, $\delta\mu$ and $[\delta, \mu]$ are the product and commutator in $\mathrm{Dist}(G)$, which lie in $\mathrm{Dist}^{+}_{p^{r+1}-1}(G)$ because of the assumptions

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.

# Structural Results

**Theorem (W. '18,'19)**

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.
- *When* $r = 0$ *we get* $U^{[0]}(G) \cong U(\mathfrak{g})$.

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.
- *When* $r = 0$ *we get* $U^{[0]}(G) \cong U(\mathfrak{g})$.
- *Each of the algebra* $U^{[r]}(G)$ *has a Poincaré-Birkhoff-Witt basis, using regular and divided powers.*

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.
- *When* $r = 0$ *we get* $U^{[0]}(G) \cong U(\mathfrak{g})$.
- *Each of the algebra* $U^{[r]}(G)$ *has a Poincaré-Birkhoff-Witt basis, using regular and divided powers.*
- *For each* $\delta \in \mathrm{Dist}^{+}_{p^r}(G)$, *the element* $\delta^{\otimes p} - \delta^{p}$ *is central, and the map* $\delta \mapsto \delta^{\otimes p} - \delta^{p}$ *is semilinear.*

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.
- *When* $r = 0$ *we get* $U^{[0]}(G) \cong U(\mathfrak{g})$.
- *Each of the algebra* $U^{[r]}(G)$ *has a Poincaré-Birkhoff-Witt basis, using regular and divided powers.*
- *For each* $\delta \in \mathrm{Dist}^{+}_{p^r}(G)$, *the element* $\delta^{\otimes p} - \delta^p$ *is central, and the map* $\delta \mapsto \delta^{\otimes p} - \delta^p$ *is semilinear.*
- *All irreducible* $U^{[r]}(G)$*-modules are finite-dimensional.*

# Structural Results

## Theorem (W. '18,'19)

- *For each* $r \in \mathbb{N}$, $U^{[r]}(G)$ *is a cocommutative Hopf algebra.*
- *For each* $r \geqslant s$, *there exists a Hopf algebra homomorphism* $U^{[r]}(G) \twoheadrightarrow U^{[s]}(G)$.
- *When* $r = 0$ *we get* $U^{[0]}(G) \cong U(\mathfrak{g})$.
- *Each of the algebra* $U^{[r]}(G)$ *has a Poincaré-Birkhoff-Witt basis, using regular and divided powers.*
- *For each* $\delta \in \mathrm{Dist}^{+}_{p^r}(G)$, *the element* $\delta^{\otimes p} - \delta^p$ *is central, and the map* $\delta \mapsto \delta^{\otimes p} - \delta^p$ *is semilinear.*
- *All irreducible* $U^{[r]}(G)$*-modules are finite-dimensional.*
- *The finite-dimensional Hopf algebra* $\mathrm{Dist}(G_r)$ *embeds inside* $U^{[r]}(G)$ *as a normal Hopf subalgebra.*

# Higher Reduced Enveloping Algebras

The algebra homomorphisms

$$U^{[r]}(G) \twoheadrightarrow U(\mathfrak{g})$$

restrict to

$$\mathrm{Dist}^+_{p^r}(G) \twoheadrightarrow \mathfrak{g}.$$

# Higher Reduced Enveloping Algebras

The algebra homomorphisms

$$U^{[r]}(G) \twoheadrightarrow U(\mathfrak{g})$$

restrict to

$$\mathrm{Dist}^+_{p^r}(G) \twoheadrightarrow \mathfrak{g}.$$

Thus, any linear form $\chi \in \mathfrak{g}^*$ can be lifted to a linear map

$$\chi : \mathrm{Dist}^+_{p^r}(G) \to \mathbb{K}.$$

# Higher Reduced Enveloping Algebras

The algebra homomorphisms

$$U^{[r]}(G) \twoheadrightarrow U(\mathfrak{g})$$

restrict to

$$\mathrm{Dist}^+_{p^r}(G) \twoheadrightarrow \mathfrak{g}.$$

Thus, any linear form $\chi \in \mathfrak{g}^*$ can be lifted to a linear map

$$\chi : \mathrm{Dist}^+_{p^r}(G) \to \mathbb{K}.$$

This allows us to define the **higher reduced enveloping algebra** associated to $\chi \in \mathfrak{g}^*$ as

$$U^{[r]}_{\chi}(G) := \frac{U^{[r]}(G)}{\langle \delta^{\otimes p} - \delta^p - \chi(\delta)^p \mid \delta \in \mathrm{Dist}^+_{p^r}(G) \rangle}.$$

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

- If $\mathfrak{g}$ has dimension $n$ then, for any $\chi \in \mathfrak{g}^*$, the $\mathbb{K}$-algebra $U_\chi^{[r]}(G)$ has dimension $p^{(r+1)n}$.

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

- If $\mathfrak{g}$ has dimension $n$ then, for any $\chi \in \mathfrak{g}^*$, the $\mathbb{K}$-algebra $U_\chi^{[r]}(G)$ has dimension $p^{(r+1)n}$.
- When $\chi = 0$, we obtain $U_0^{[r]}(G) = \mathrm{Dist}(G_{r+1})$.

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

- If $\mathfrak{g}$ has dimension $n$ then, for any $\chi \in \mathfrak{g}^*$, the $\mathbb{K}$-algebra $U_\chi^{[r]}(G)$ has dimension $p^{(r+1)n}$.
- When $\chi = 0$, we obtain $U_0^{[r]}(G) = \mathrm{Dist}(G_{r+1})$.
- $\mathrm{Dist}(G_r)$ lies inside $U_\chi^{[r]}(G)$ for all $\chi \in \mathfrak{g}^*$.

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

- If $\mathfrak{g}$ has dimension $n$ then, for any $\chi \in \mathfrak{g}^*$, the $\mathbb{K}$-algebra $U_\chi^{[r]}(G)$ has dimension $p^{(r+1)n}$.
- When $\chi = 0$, we obtain $U_0^{[r]}(G) = \mathrm{Dist}(G_{r+1})$.
- $\mathrm{Dist}(G_r)$ lies inside $U_\chi^{[r]}(G)$ for all $\chi \in \mathfrak{g}^*$.
- Every irreducible $U^{[r]}(G)$-module is a $U_\chi^{[r]}(G)$-modules for some $\chi \in \mathfrak{g}^*$.

# Higher Reduced Enveloping Algebras

These algebras have a number of nice properties:

- If $\mathfrak{g}$ has dimension $n$ then, for any $\chi \in \mathfrak{g}^*$, the $\mathbb{K}$-algebra $U_\chi^{[r]}(G)$ has dimension $p^{(r+1)n}$.
- When $\chi = 0$, we obtain $U_0^{[r]}(G) = \mathrm{Dist}(G_{r+1})$.
- $\mathrm{Dist}(G_r)$ lies inside $U_\chi^{[r]}(G)$ for all $\chi \in \mathfrak{g}^*$.
- Every irreducible $U^{[r]}(G)$-module is a $U_\chi^{[r]}(G)$-modules for some $\chi \in \mathfrak{g}^*$.
- Given $g \in G$, $U_\chi^{[r]}(G) \cong U_{g\cdot\chi}^{[r]}(G)$, where $g \cdot \chi$ is the twisted coadjoint action.

# Representation Theoretic Results

When $G$ is a semisimple simply-connected algebraic group, we can obtain a Steinberg decomposition.

# Representation Theoretic Results

When $G$ is a semisimple simply-connected algebraic group, we can obtain a Steinberg decomposition.

**Theorem (W. '19)**

*There is a bijection*

$$\Psi : \mathrm{Irr}(U^{[r]}(G)) \to \mathrm{Irr}(\mathrm{Dist}(G_r)) \times \mathrm{Irr}(U(\mathfrak{g}))$$

*which sends $M$ to $(P, \mathrm{Hom}_{G_r}(P, M))$, where $P$ is the unique irreducible $\mathrm{Dist}(G_r)$-submodule of $M$. Furthermore, the reverse map sends $(P, N)$ to the $U^{[r]}(G)$-module $(U^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U(\mathfrak{g})} N = P \otimes_{\mathbb{K}} N$.*

# Representation Theoretic Results

When $G$ is a semisimple simply-connected algebraic group, we can obtain a Steinberg decomposition.

**Theorem (W. '19)**

*There is a bijection*

$$\Psi : \mathrm{Irr}(U^{[r]}(G)) \to \mathrm{Irr}(\mathrm{Dist}(G_r)) \times \mathrm{Irr}(U(\mathfrak{g}))$$

*which sends $M$ to $(P, \mathrm{Hom}_{G_r}(P, M))$, where $P$ is the unique irreducible $\mathrm{Dist}(G_r)$-submodule of $M$. Furthermore, the reverse map sends $(P, N)$ to the $U^{[r]}(G)$-module $(U^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U(\mathfrak{g})} N = P \otimes_{\mathbb{K}} N$.*

**Theorem (W. '19)**

*Let $\chi \in \mathfrak{g}^*$. The above map restricts to a bijection*

$$\Psi_\chi : \mathrm{Irr}(U_\chi^{[r]}(G)) \to \mathrm{Irr}(\mathrm{Dist}(G_r)) \times \mathrm{Irr}(U_\chi(\mathfrak{g})).$$

## Consequences

Given $P \in \mathrm{Irr}(\mathrm{Dist}(G_r))$, $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and

$$\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$$

we define the **teenage Verma module**

$$Z_\chi^{[r]}(P, \lambda) := (U_\chi^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U_\chi(\mathfrak{g})} Z_\chi(\lambda) \cong P \otimes_{\mathbb{K}} Z_\chi(\lambda).$$

# Consequences

Given $P \in \mathrm{Irr}(\mathrm{Dist}(G_r))$, $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and

$$\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$$

we define the **teenage Verma module**

$$Z_\chi^{[r]}(P, \lambda) := (U_\chi^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U_\chi(\mathfrak{g})} Z_\chi(\lambda) \cong P \otimes_{\mathbb{K}} Z_\chi(\lambda).$$

Under certain assumptions, we obtain the following properties.

# Consequences

Given $P \in \mathrm{Irr}(\mathrm{Dist}(G_r))$, $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and

$$\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$$

we define the **teenage Verma module**

$$Z_\chi^{[r]}(P, \lambda) := (U_\chi^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U_\chi(\mathfrak{g})} Z_\chi(\lambda) \cong P \otimes_{\mathbb{K}} Z_\chi(\lambda).$$

Under certain assumptions, we obtain the following properties.

- Every irreducible $U_\chi^{[r]}(G)$-module is a homomorphic image of a teenage Verma module.

# Consequences

Given $P \in \mathrm{Irr}(\mathrm{Dist}(G_r))$, $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and

$$\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$$

we define the **teenage Verma module**

$$Z_\chi^{[r]}(P, \lambda) := (U_\chi^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U_\chi(\mathfrak{g})} Z_\chi(\lambda) \cong P \otimes_{\mathbb{K}} Z_\chi(\lambda).$$

Under certain assumptions, we obtain the following properties.

- Every irreducible $U_\chi^{[r]}(G)$-module is a homomorphic image of a teenage Verma module.
- If $\chi$ is regular (i.e. $\dim(C_G(\chi)) = \dim(\mathfrak{h})$), then each $Z_\chi^{[r]}(P, \lambda)$ is an irreducible $U^{[r]}(G)$-module.

# Consequences

Given $P \in \mathrm{Irr}(\mathrm{Dist}(G_r))$, $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$, and

$$\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$$

we define the **teenage Verma module**

$$Z_\chi^{[r]}(P, \lambda) := (U_\chi^{[r]}(G) \otimes_{\mathrm{Dist}(G_r)} P) \otimes_{U_\chi(\mathfrak{g})} Z_\chi(\lambda) \cong P \otimes_{\mathbb{K}} Z_\chi(\lambda).$$

Under certain assumptions, we obtain the following properties.

- Every irreducible $U_\chi^{[r]}(G)$-module is a homomorphic image of a teenage Verma module.
- If $\chi$ is regular (i.e. $\dim(C_G(\chi)) = \dim(\mathfrak{h})$), then each $Z_\chi^{[r]}(P, \lambda)$ is an irreducible $U^{[r]}(G)$-module.
- Each teenage Verma module $Z_\chi^{[r]}(P, \lambda)$ has dimension $p^{\dim(\mathfrak{n}^-)} \dim(P)$, so the maximal dimension of an irreducible $U^{[r]}(G)$-module is $p^{(r+1)\dim(\mathfrak{n}^-)}$, obtained via the Steinberg module.

# Thank you for listening!